

SYZYGIES OF COHEN-MACAULAY MODULES AND GROTHENDIECK GROUPS

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ABSTRACT. We study the converse of a theorem of Butler and Auslander-Reiten. We show that a Cohen-Macaulay local ring with an isolated singularity has only finitely many isomorphism classes of indecomposable summands of syzygies of Cohen-Macaulay modules if the Auslander-Reiten sequences generate the relation of the Grothendieck group of finitely generated modules. This extends a recent result of Hiramatsu, which gives an affirmative answer in the Gorenstein case to a conjecture of Auslander.

1. INTRODUCTION

Throughout this note, let (R, \mathfrak{m}, k) be a Cohen-Macaulay local ring with an isolated singularity. We denote by $\mathbf{CM}(R)$ (resp. $\mathbf{mod}(R)$) the category of (maximal) Cohen-Macaulay R -modules (resp. finitely generated R -modules) with R -homomorphisms.

Let $\mathbf{G}(\mathbf{CM}(R))$ be the quotient of the free abelian group $\bigoplus \mathbb{Z}[X]$ generated by the isomorphism classes $[X]$ of modules X in $\mathbf{CM}(R)$ by the subgroup generated by

$$\{[X] + [Z] - [Y] \mid Y \cong X \oplus Z\}.$$

Thus $\mathbf{G}(\mathbf{CM}(R))$ is isomorphic to the free abelian group generated by the isomorphism classes of indecomposable Cohen-Macaulay R -modules.

We denote by $\mathbf{Ex}(\mathbf{CM}(R))$ the subgroup of $\mathbf{G}(\mathbf{CM}(R))$ generated by

$$\{[X] + [Z] - [Y] \mid \text{there exists an exact sequence } 0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \text{ in } \mathbf{CM}(R)\}.$$

Then the quotient group $\mathbf{G}(\mathbf{CM}(R))/\mathbf{Ex}(\mathbf{CM}(R))$ is nothing but the Grothendieck group $\mathbf{K}_0(\mathbf{CM}(R))$ of $\mathbf{CM}(R)$ and therefore coincides with the Grothendieck group of $\mathbf{mod}(R)$.

We also denote by $\mathbf{AR}(\mathbf{CM}(R))$ the subgroup of $\mathbf{G}(\mathbf{CM}(R))$ generated by

$$\left\{ [X] + [Z] - [Y] \mid \begin{array}{l} \text{there exists an Auslander-Reiten sequence} \\ 0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \text{ in } \mathbf{CM}(R) \end{array} \right\}.$$

Concerning the relationship between $\mathbf{Ex}(\mathbf{CM}(R))$ and $\mathbf{AR}(\mathbf{CM}(R))$, the following theorem holds; see [5], [3, Proposition 2.2] and [10, Theorem 13.7].

Theorem 1.1 (Butler, Auslander-Reiten). *If R is of finite CM type, then $\mathbf{Ex}(\mathbf{CM}(R)) = \mathbf{AR}(\mathbf{CM}(R))$.*

Here we say that R is of *finite CM type* if there are only finitely many isomorphism classes of indecomposable Cohen-Macaulay R -modules.

Auslander conjectured that the converse of Theorem 1.1 holds. Our main result is the following theorem, which yields a weaker version of the converse of Theorem 1.1.

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Theorem 1.2. *If $\text{Ex}(\text{CM}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{AR}(\text{CM}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}$, then there exist only finitely many isomorphism classes of indecomposable summands of (first) syzygies of Cohen-Macaulay R -modules.*

If R is Gorenstein, then every Cohen-Macaulay R -module is a first syzygy of some Cohen-Macaulay R -module. Hence Theorem 1.2 recovers the following result, which is proved by Hiramatsu [7] and gives an affirmative answer to Auslander's conjecture in the case of Gorenstein local rings.

Corollary 1.3 (Hiramatsu). *Assume that R is Gorenstein. If $\text{Ex}(\text{CM}(R)) = \text{AR}(\text{CM}(R))$, then R is of finite CM type.*

When R is a two dimensional complete local ring and k is algebraically closed, it is shown in [6, Corollary 3.3] that R has a finite number of isomorphism classes of indecomposable summands of syzygies of Cohen-Macaulay modules if and only if R is a rational singularity. This fact provides the following corollary.

Corollary 1.4. *Assume that R is a complete local ring of dimension two with k algebraically closed. If $\text{Ex}(\text{CM}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{AR}(\text{CM}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}$, then R is a rational singularity.*

In the rest of this note, we give a proof of Theorem 1.2.

2. PROOF OF OUR THEOREM

As in the introduction, we always assume that (R, \mathfrak{m}, k) is a Cohen-Macaulay local ring with an isolated singularity. All R -modules are assumed to be finitely generated.

We denote by $\underline{\text{mod}}(R)$ (resp. $\underline{\text{CM}}(R)$) the stable category of $\text{mod}(R)$ (resp. $\text{CM}(R)$). These categories are defined in such a way that the objects are the same as those of $\text{mod}(R)$ (resp. $\text{CM}(R)$), and for objects M, N , the set of morphisms from M to N is $\underline{\text{Hom}}_R(M, N)$, defined to be the quotient of $\text{Hom}_R(M, N)$ by the R -submodule consisting of homomorphisms factoring through free R -modules.

To give a proof of Theorem 1.2, we prepare several lemmas. The first one is given in [7, Lemma 2.1].

Lemma 2.1. *There exists a Cohen-Macaulay R -module X such that for any non-free Cohen-Macaulay R -module M one has $\underline{\text{Hom}}_R(M, X) \neq 0$.*

We denote by ΩM the first syzygy of R -module M , and by $\text{Tr}M$ the (Auslander) transpose of M ; see [10, Definition (3.5)]. The modules ΩM and Tr are uniquely determined by M up to free summands.

We have the lemma below; see [9, Proposition 2.7] for instance.

Lemma 2.2. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence in $\text{mod}(R)$ and U an R -module. Then there exists a long exact sequence*

$$\begin{aligned} \underline{\text{Hom}}_R(U, \Omega A) &\rightarrow \underline{\text{Hom}}_R(U, \Omega B) \rightarrow \underline{\text{Hom}}_R(U, \Omega C) \rightarrow \\ \underline{\text{Hom}}_R(U, A) &\rightarrow \underline{\text{Hom}}_R(U, B) \rightarrow \underline{\text{Hom}}_R(U, C). \end{aligned}$$

Here, we define $\Omega^{-1}M$ to be the module $\text{Tr}\Omega\text{Tr}M$ for an R -module M . The assignments $M \mapsto \Omega M$, $M \mapsto \text{Tr}M$ and $M \mapsto \Omega^{-1}M$ define additive endofunctors of $\underline{\text{mod}}(R)$. Moreover, the following lemma holds; see [4, Corollary 3.3].

Lemma 2.3. *Let X, Y be R -modules. Then there is an isomorphism*

$$\underline{\mathrm{Hom}}_R(X, \Omega Y) \rightarrow \underline{\mathrm{Hom}}_R(\Omega^{-1}X, Y),$$

which is natural in X, Y (i.e. one has an adjoint pair $(\Omega^{-1}, \Omega) : \underline{\mathrm{mod}}(R) \rightarrow \underline{\mathrm{mod}}(R)$.)

We denote by $\Omega\mathrm{CM}(R)$ the full subcategory of $\underline{\mathrm{mod}}(R)$ consisting of first syzygies of Cohen-Macaulay R -modules.

Remark 2.4. It is easy to see that $\Omega\mathrm{CM}(R)$ is closed under direct summands. In particular, the following are equivalent.

- (1) There are only finitely many non-isomorphic indecomposable modules in $\Omega\mathrm{CM}(R)$.
- (2) There are only finitely many non-isomorphic indecomposable summands of modules in $\Omega\mathrm{CM}(R)$.

If one/both of these conditions is/are satisfied, we say that R is of *finite $\Omega\mathrm{CM}$ type*.

Now we give some properties of modules in $\Omega\mathrm{CM}(R)$.

Lemma 2.5. *If $M \in \Omega\mathrm{CM}(R)$, then $\Omega^{-1}M$ is in $\mathrm{CM}(R)$ and $M \cong \Omega\Omega^{-1}M$ up to free summands.*

Proof. Since M is in $\Omega\mathrm{CM}(R)$, there is a Cohen-Macaulay module N such that M is the first syzygy of N . By the proof of [2, Proposition 2.21], there is an exact sequence $0 \rightarrow R^{\oplus a} \rightarrow \Omega^{-1}M \oplus R^{\oplus b} \rightarrow N \rightarrow 0$ with some integers a, b . This implies that $\Omega^{-1}M$ is a Cohen-Macaulay module. As M is a syzygy, M is isomorphic to $\Omega\Omega^{-1}M = \Omega\mathrm{Tr}\Omega\mathrm{Tr}M$ up to free summands by [2, Theorem 2.17]. \blacksquare

Next we investigate the non-free part of a given module.

Lemma 2.6. *Let M be a finitely generated R -module, and \widehat{R} the completion of R .*

- (1) *M has an R -free summand if and only if $M \otimes_R \widehat{R}$ has a \widehat{R} -free summand.*
- (2) *There is a unique decomposition $M \cong \underline{M} \oplus F$ of M up to isomorphism with F free such that \underline{M} has no free summands. We call this module \underline{M} the non-free part of M .*
- (3) *Let A, B be finitely generated R -modules. If A is a direct summand of B , then \underline{A} is a direct summand of \underline{B} .*

Proof. (1) The assertion follows from [8, Corollary 1.15 (i)].

(2) We can take a maximal free summand $R^{\oplus a}$ of M to have a decomposition $M \cong M' \oplus R^{\oplus a}$ where M' has no free summands. Suppose that there is another decomposition $M \cong M'' \oplus R^{\oplus b}$ where M'' has no free summands. Taking the completion, we have $M \otimes_R \widehat{R} \cong (M' \otimes_R \widehat{R}) \oplus \widehat{R}^{\oplus a} \cong (M'' \otimes_R \widehat{R}) \oplus \widehat{R}^{\oplus b}$. By (1), $M' \otimes_R \widehat{R}$ and $M'' \otimes_R \widehat{R}$ have no free summands. Since the Krull-Schmidt property holds over \widehat{R} , we have $M' \otimes_R \widehat{R} \cong M'' \otimes_R \widehat{R}$ and $a = b$. Using [8, Corollary 1.15 (ii)], we have $M' \cong M''$.

(3) Suppose that A is a direct summand of B . Then \underline{A} is also a direct summand of \underline{B} . Hence we have a decomposition $B \cong \underline{A} \oplus C$. It follows from (2) that the non-free part \underline{B} of B is isomorphic to the module $\underline{A} \oplus \underline{C}$. In particular, \underline{B} has \underline{A} as a direct summand. \blacksquare

Since there is an isomorphism $\underline{\mathrm{Hom}}_R(M, N) \cong \mathrm{Tor}_1^R(\mathrm{Tr}(M), N)$ for finitely generated R -modules M, N (see [10, Lemma (3.9)]) and since we assume that R is an isolated singularity, we can show that the length of the R -module $\underline{\mathrm{Hom}}_R(M, N)$ is finite for any

M, N in $\mathbf{CM}(R)$. We denote by $[M, N]$ the integer $\text{length}_R(\underline{\text{Hom}}_R(M, N))$. The following proposition plays a key role in the proof of our theorem. For the definition and basic properties of an Auslander-Reiten sequence, we refer the reader to [8, 10].

Proposition 2.7. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an Auslander-Reiten sequence, and U be a non-free Cohen-Macaulay R -module. Then the following hold.*

- (1) *The induced sequence $\underline{\text{Hom}}_R(U, B) \rightarrow \underline{\text{Hom}}_R(U, C) \rightarrow 0$ is exact if and only if C is not a direct summand of U .*
- (2) *Suppose that U is an indecomposable module in $\Omega\mathbf{CM}(R)$.*
 - (a) *If U is not isomorphic to the non-free part of ΩC , then the induced sequence $0 \rightarrow \underline{\text{Hom}}_R(U, A) \rightarrow \underline{\text{Hom}}_R(U, B)$ is exact.*
 - (b) *If $[U, C] + [U, A] - [U, B] \neq 0$, then U is isomorphic to either C or the non-free part of ΩC .*

Proof. (1) Assume that C is not a direct summand of U . Using the lifting property of an Auslander-Reiten sequence, every homomorphism from U to C factors through the map $B \rightarrow C$. This means that $\underline{\text{Hom}}_R(U, B) \rightarrow \underline{\text{Hom}}_R(U, C)$ is surjective. Conversely, suppose that C is a direct summand of U . Then there is a split epimorphism $f : U \rightarrow C$. Let $g : C \rightarrow U$ be the right-inverse of f . If $\underline{\text{Hom}}_R(U, B) \rightarrow \underline{\text{Hom}}_R(U, C)$ is surjective, then so is the morphism $\text{Hom}_R(U, B) \rightarrow \text{Hom}_R(U, C)$ because of the commutative diagram

$$\begin{array}{ccccc}
 \text{Hom}_R(U, B) & \longrightarrow & \text{Hom}_R(U, C) & \longrightarrow & \text{Ext}_R^1(U, A) \\
 \downarrow & & \downarrow & & \parallel \\
 \underline{\text{Hom}}_R(U, B) & \longrightarrow & \underline{\text{Hom}}_R(U, C) & \longrightarrow & \text{Ext}_R^1(U, A) \\
 \downarrow & & \downarrow & & \\
 0 & & 0 & &
 \end{array}$$

with exact rows and columns; see the proof of [9, Proposition 2.7 (2)]. Hence there is a lift $h : U \rightarrow B$ of f . The morphism $hg : C \rightarrow B$ is the right-inverse of the morphism $B \rightarrow C$. This contradicts the definition of the Auslander-Reiten sequences.

(2a) Let K be the kernel of the map $\underline{\text{Hom}}_R(U, A) \rightarrow \underline{\text{Hom}}_R(U, B)$. By Lemma 2.2, the sequence $\underline{\text{Hom}}_R(U, \Omega B) \rightarrow \underline{\text{Hom}}_R(U, \Omega C) \rightarrow K \rightarrow 0$ is exact. Then the equality $K = 0$ is equivalent to the exactness of $\underline{\text{Hom}}_R(\Omega^{-1}U, B) \rightarrow \underline{\text{Hom}}_R(\Omega^{-1}U, C) \rightarrow 0$, by using Lemma 2.3. Lemma 2.5 implies that $\Omega^{-1}U$ is Cohen-Macaulay. By (1), K is zero if and only if C is not a direct summand of $\Omega^{-1}U$.

Suppose that C is a direct summand of $\Omega^{-1}U$. By Lemma 2.5 there is a free R -module F such that ΩC is a direct summand of $U \oplus F$. Lemma 2.6 (3) implies that the non-free part of ΩC is a direct summand of U , and is isomorphic to U as U is indecomposable.

(2b) Let L be the cokernel of the map $\underline{\text{Hom}}_R(U, B) \rightarrow \underline{\text{Hom}}_R(U, C)$. Then we have an equality $[U, C] + [U, A] - [U, B] = \text{length}_R(L) + \text{length}_R(K)$. By (1), $\text{length}_R(L) \neq 0$ implies that U is isomorphic to C . By (2a), $\text{length}_R(K) \neq 0$ implies that U is isomorphic to the non-free part of ΩC . ■

Now we can give a proof of our theorem.

Proof of Theorem 1.2. Let X be the module that satisfies the conditions in Lemma 2.1. Then there is an exact sequence with a free module P :

$$0 \rightarrow \Omega X \rightarrow P \rightarrow X \rightarrow 0.$$

Since $\text{Ex}(\text{CM}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{AR}(\text{CM}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}$, there are a finite number of indecomposable Cohen-Macaulay R -modules C_1, \dots, C_n and an equality in $\text{G}(\text{CM}(R))$:

$$a([X] + [\Omega X] - [P]) = \sum_{i=1}^n b_i([A_i] + [C_i] - [B_i]),$$

where a is a positive integer, b_i are integers and $[A_i] + [C_i] - [B_i]$ come from Auslander-Reiten sequences $0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$. We have an equality in \mathbb{Z} :

$$(2.7.1) \quad a[U, X \oplus \Omega X] = \sum_{i=1}^n b_i([U, A_i] + [U, C_i] - [U, B_i])$$

for each non-free indecomposable module U in $\Omega\text{CM}(R)$, since in general the equality $[U, \bigoplus_{s=1}^t L_s] = \sum_{s=1}^t [U, L_s]$ holds for any R -modules L_1, \dots, L_t . The left-hand side of (2.7.1) is nonzero by the choice of X in Lemma 2.1, and hence so is the right-hand side. By Proposition 2.7 (2b), this can occur only when U is isomorphic to either C_i or the non-free part of ΩC_i for some i , and we conclude that the number of isomorphism classes of such modules U is finite. \blacksquare

Remark 2.8. The converse of Theorem 1.1 has been proved by Auslander [1] for artin algebras and by Auslander-Reiten [4] for one dimensional complete local domains. We shall give examples of finite dimensional local algebras and one dimensional complete local domains which are of finite ΩCM type but not of finite CM type. Thus finite ΩCM type is not sufficient to hold the equality $\text{EX}(\text{CM}(R)) = \text{AR}(\text{CM}(R))$, and the converse of Theorem 1.2 is not true in general if we replace the condition $\text{EX}(\text{CM}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{AR}(\text{CM}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the condition $\text{EX}(\text{CM}(R)) = \text{AR}(\text{CM}(R))$.

Example 2.9. Let $R = k[X, Y]/(X, Y)^2$ with k a field. Then R is a finite dimensional local k -algebra and not of finite CM type. Since the first syzygy M of a non-free R -module is a submodule of a direct sum of copies of the maximal ideal \mathfrak{m} , the module M is annihilated by $\text{ann}(\mathfrak{m}) = \mathfrak{m}$. So M is a module over R/\mathfrak{m} . In particular, every non-free indecomposable module in $\Omega\text{CM}(R)$ is isomorphic to R/\mathfrak{m} , and R is of finite ΩCM type.

Example 2.10. Let $S = k[[t]]$ with k a field, $n \geq 1$ be an integer and $R = k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$ be the subring of S . Then R is a one dimensional complete local domain and the maximal ideal $\mathfrak{m} = t^n S$ of R is isomorphic to S as an R -module. Let M be a non-free indecomposable module in $\Omega\text{CM}(R)$. We show that M can be regarded as an S -submodule of some free S -module. In fact, there exist a Cohen-Macaulay R -module N and a short exact sequence $0 \rightarrow M \rightarrow R^{\oplus a} \rightarrow N \rightarrow 0$ coming from a minimal free resolution of N . By the minimality, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathfrak{m}^{\oplus a} & \longrightarrow & L \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & R^{\oplus a} & \longrightarrow & N \longrightarrow 0. \end{array}$$

By the snake lemma, L is viewed as a submodule of N and thus a Cohen-Macaulay R -module. Replacing \mathfrak{m} with S and multiplying t^n , we get a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M & \longrightarrow & S^{\oplus a} & \longrightarrow & L \longrightarrow 0 \\
& & \downarrow t^n & & \downarrow t^n & & \downarrow t^n \\
0 & \longrightarrow & M & \longrightarrow & S^{\oplus a} & \longrightarrow & L \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & M/t^n M & \longrightarrow & S^{\oplus a}/t^n S^{\oplus a} & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

where the rows and columns are both exact. Applying the snake lemma again, we see that the morphism $M/t^n M \rightarrow S^{\oplus a}/t^n S^{\oplus a}$ in the diagram above is injective, as L is a Cohen-Macaulay R -module. Since t^{n+1} annihilates $S^{\oplus a}/t^n S^{\oplus a}$, it also annihilates $M/t^n M$. Hence $t^{n+1}M \subset t^n M$. Identifying M as an R -submodule of $S^{\oplus a}$, we observe $tM \subset M$, which makes M be an S -submodule of $S^{\oplus a}$. Since S is a discrete valuation ring, the submodule M of the free S -module $S^{\oplus a}$ is free. This shows that the nonisomorphic indecomposable R -modules in $\Omega\text{CM}(R)$ are R and $S(\cong \mathfrak{m})$, which especially says that R is of finite ΩCM type. On the other hand, R is not of finite CM type when $n \geq 4$ (see [8, Theorem 4.10]).

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